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The extended metamorphosis of a complete bipartite design into a cycle system

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Dedicated to Curt Lindner on the occasion of his 65th birthday

Abstract

A $K_{t,t}$-design of order n is an edge-disjoint decomposition of K_n into copies of $K_{t,t}$. When t is odd, an extended metamorphosis of a $K_{t,t}$-design of order n into a $2t$-cycle system of order n is obtained by taking $(t-1)/2$ edge-disjoint cycles of length $2t$ from each $K_{t,t}$ block, and rearranging all the remaining 1-factors in each $K_{t,t}$ block into further $2t$-cycles. The 'extended' refers to the fact that as many subgraphs isomorphic to a $2t$-cycle as possible are removed from each $K_{t,t}$ block, rather than merely one subgraph.

In this paper an extended metamorphosis of a $K_{t,t}$-design of order congruent to $1 \pmod{4t^2}$ into a $2t$-cycle system of the same order is given for all odd $t > 3$. A metamorphosis of a 2-fold $K_{t,t}$-design of any order congruent to $1 \pmod{t^2}$ into a $2t$-cycle system of the same order is also given, for all odd $t > 3$. (The case $t = 3$ appeared in Ars Combin. 64 (2002) 65–80.)

When t is even, the graph $K_{t,t}$ is easily seen to contain $t/2$ edge-disjoint cycles of length $2t$, and so the metamorphosis in that case is straightforward.

1. Introduction

The concept of a *metamorphosis* of a design was introduced by Lindner relatively recently. There has been a spate of papers in the area, as evidenced by the references included here.

We start with some definitions. A *G-design* of a graph H is an edge-disjoint decomposition of H into isomorphic copies of G. In the case that H is a complete graph K_n, we refer to such a G-design of K_n as a G-design of order n. If H is the graph λK_n, with λ edges between each pair of vertices, then the G-design is a λ-fold G-design of order n. Moreover, if we let the vertex set of K_n be V, and the collection of isomorphic copies of G be B, then the notation (V,B) is also used to denote a G-design of order n.

In this paper we shall chiefly be concerned with complete bipartite designs $K_{t,t}$ and G-designs where G is a cycle; this latter type of G-design where G is a cycle is more commonly referred to as a *cycle system*. We assume the reader knows what a complete bipartite graph, a 1-factor and a cycle are. Our notation for these is as follows.

The complete bipartite graph with vertex partition $A \cup B$ will be denoted by $\{A : B\}$, where the elements of A and B can of course be listed in any order. When there is no confusion, an edge with endpoints x and y will be denoted xy or yx for short; otherwise $\{x, y\}$ will be used.

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We shall frequently need to describe explicit 1-factors in a complete bipartite graph $K_{t,t}$; following the notation in [4], if the edges of a 1-factor in $K_{t,t} = \{1,2,\ldots,t : 1',2',\ldots,t'\}$ are $\{1,1'\},\{2,2'\},\ldots,\{t,t'\}$, we shall write $[1,2,\ldots,t : 1',2',\ldots,t']$ to denote this 1-factor.

The cycle with vertex set $\{a_1,a_2,\ldots,a_n\}$ and edge set $\{a_ia_{i+1} \mid 1 \leqslant i \leqslant n-1\} \cup \{a_1a_n\}$ will be denoted by $(a_1,a_2,\ldots,a_n)$ or $(a_1,a_n,\ldots,a_3,a_2)$ or any cyclic shift of these.

Suppose we have a graph G, with a proper subgraph G'. Let the complement of G' in G be denoted by G''. Also suppose that n is an admissible order of both a G-design and a G'-design. We describe a metamorphosis of a G-design of order n into a G'-design of order n as follows. Begin with a G-design of order n. From each copy of G in the design (which we may refer to as a *G-block*), retain a copy of the subgraph G'. Then all the edges from the complementary parts isomorphic to G'' are taken and rearranged into further copies of G'. Thus a G' design is obtained from a G-design, with each G-block giving rise to a G'-block (which is a subgraph of the original block), and with further G'-blocks arising from precisely the edges remaining from all the copies of the complement G''.

Several recent papers have dealt with metamorphoses of (λ-fold) G-designs into G'-designs. When G is K_4, metamorphoses into: C_3 ([8], and in part [5]); C_3 plus a pendant edge [6]; C_4 [10]; K_4 minus one edge [9,7], have all been shown to exist for appropriate orders. When G is a 4-wheel (that is, a simple graph with five vertices consisting of a 4-cycle with a fifth vertex—the centre—of degree 4, adjacent to the four vertices in the 4-cycle), metamorphoses into bowties (two triangles sharing a common vertex) [2] and also into 4-cycles [3] have been constructed. Furthermore, Adams et al. [1] consider *simultaneous* metamorphoses of small k-wheel systems for $k = 3,4,6$.

In [4], a metamorphosis of a $K_{3,3}$-design into a 6-cycle system was given for each admissible order. Moreover, the same problem was dealt with in the case of a λ-fold design, when each edge of the complete graph K_n is taken λ times.

Note that of course a $K_{2,2}$-design is already a 4-cycle system, and in fact any $K_{2m,2m}$-design may also be regarded as a $4m$-cycle system, because, as is well-known, each copy of $K_{2m,2m}$ has an easy edge-disjoint decomposition into m copies of a $4m$-cycle. One such, with $K_{2m,2m} = \{i \mid 1 \leqslant i \leqslant 2m\} : \{i' \mid 1 \leqslant i \leqslant 2m\}\}$, is

$$(1,(2j+1)',2,(2j+2)',3,(2j+3)',\ldots,2m-1,(2j+2m-1)',2m,(2j+2m)')$$

for $1 \leqslant j \leqslant m$ (entries modulo $2m$).

The case of a metamorphosis of a $K_{2m+1,2m+1}$-design into a $(4m+2)$-cycle system remains, for $m > 1$. (The paper [4] covers the case $m = 1$.) Here we really consider an *extended* metamorphosis of a $K_{2m+1,2m+1}$-design, retaining not *one* subgraph isomorphic to a $(4m+2)$-cycle, but m such subgraphs, and rearranging the remaining edges (a 1-factor of $K_{2m+1,2m+1}$) for each of the $K_{2m+1,2m+1}$ blocks, into further $(4m+2)$-cycles.

In what follows we give an extended metamorphosis of a $K_{2m+1,2m+1}$-design of any order congruent to 1 (modulo $4(2m+1)^2$) into a $(4m+2)$-cycle system of the same order, for all $m \geqslant 2$. (The case $m = 1$ appears in [4].) We also give an extended metamorphosis of a 2-fold $K_{2m+1,2m+1}$-design of any order congruent to 1 (modulo $(2m+1)^2$). (These of course then give a λ-fold metamorphosis of the same order for any odd or even λ, respectively.)

2. Necessary conditions

Since $K_{2m+1,2m+1}$ is regular of degree $2m+1$, the λ-fold complete graph λK_n must have its degree, $\lambda(n-1)$, divisible by $2m+1$ for a λ-fold $K_{2m+1,2m+1}$-design of order n to exist, and furthermore the number of edges in λK_n, namely $\lambda n(n-1)/2$, must be divisible by the number of edges in $K_{2m+1,2m+1}$, namely $(2m+1)^2$. These requirements lead to the following table of (partial) expected spectrum values.

$K_{2m+1,2m+1}$-design

λ	Order n includes:
1 or odd, coprime to $2m+1$	$1 \pmod{(2m+1)^2}$
2 or even	$1 \pmod{(2m+1)^2}$
$2m+1$	$0,\ 1 \pmod{(2m+1)}$
$(2m+1)^2$	Any order, at least $4m+2$

We also record necessary conditions for existence of a λ-fold $(4m+2)$-cycle system.

$(4m+2)$-cycle system

λ	Order n includes:
1 or odd, coprime to $2m+1$	$1\,(\mathrm{mod}\,4(2m+1))$
2 or even	$0,\ 1\,(\mathrm{mod}\,2m+1)$
$2m+1$	$1\,(\mathrm{mod}\,4)$
$4m+2$	Any order, at least $4m+2$

Combining the above conditions, we see that some conditions for existence of an extended metamorphosis from a λ-fold $K_{2m+1,2m+1}$-design to a λ-fold $(4m+2)$-cycle system include the following:

λ	Order n $(n \geqslant 4m+2)$ includes:
1	$1\,(\mathrm{mod}\,4(2m+1)^2)$
2	$1\,(\mathrm{mod}\,(2m+1)^2)$
$2m+1$	$1\,(\mathrm{mod}\,4(2m+1))$
$4m+2$	$0,\ 1\,(\mathrm{mod}\,(2m+1))$
$(2m+1)^2$	$1\,(\mathrm{mod}\,4)$
$2(2m+1)^2$	Any order, at least $4m+2$

In the remainder of this paper we construct an extended metamorphosis in the case $\lambda=1$ for all orders congruent to 1 (modulo $4(2m+1)^2$), and in the case $\lambda=2$ for all orders congruent to 1 (modulo $(2m+1)^2$).

3. Some necessary decompositions: $\lambda = 1$

The following result is doubtless well-known. We include specific details for completeness.

Lemma 3.1. *There exists a decomposition of* $K_{2m+1,2m+1}$ *into* m *cycles of length* $4m+2$ *and one* 1-*factor*.

Proof. Let $K_{2m+1,2m+1}$ have vertex set $\{i \mid 0 \leqslant i \leqslant 2m\} \cup \{i' \mid 0 \leqslant i \leqslant 2m\}$. Then a suitable decomposition is $\{\{i,i'\} \mid 0 \leqslant i \leqslant 2m\}$ (the 1-factor), and

$$(0,(2m+1-i)',1,(1-i)',2,(2-i)',3,(3-i)',\ldots,(2m)',i,0',\ldots,2m,(2m-i)')$$

for $i = 1,3,\ldots,2m-1$ (all odd i), entries modulo $2m+1$. □

Corollary 3.2. *There exists a decomposition of* $K_{8m+4,8m+4}$ *into* $16m$ *cycles of length* $4m+2$ *and a subgraph of* $K_{8m+4,8m+4}$ *which is regular of degree* 4.

Proof. Applying Lemma 3.1 with four sets of $2m+1$ vertices in each half of the vertex set of $K_{8m+4,8m+4}$, and so with 16 sets of $K_{2m+1,2m+1}$, yields the result. □

Lemma 3.3. *There exists a decomposition of* $K_{8m+4,8m+4}$ *into copies of* $K_{2m+1,2m+1}$, *which has an extended metamorphosis into* $16m+8$ *cycles of length* $4m+2$.

Proof. Let the vertex set of $K_{8m+4,8m+4}$ be

$$\{a_i,b_i,c_i,d_i \mid 0 \leqslant i \leqslant 2m\} \cup \{A_i,B_i,C_i,D_i \mid 0 \leqslant i \leqslant 2m\}.$$

Take the obvious decomposition of this into sixteen $K_{2m+1,2m+1}$-blocks:

$$\{\{x_i \mid 0 \leqslant i \leqslant 2m\} : \{Y_i \mid 0 \leqslant i \leqslant 2m\}\} \quad \text{for } x=a,b,c,d,\ Y=A,B,C,D.$$

From each $K_{2m+1,2m+1}$-block we can choose a 1-factor, which remains when m $(4m+2)$-cycles are removed (see Lemma 3.1). We rearrange these sixteen 1-factors into eight $(4m+2)$-cycles. We give these in the form of two starter cycles, with fixed subscripts, but with the entries to be cycled according to the permutation $(a\,b\,c\,d)(A\,B\,C\,D)$, thus yielding eight cycles altogether.

The case $m=1$ appeared in [4]; for clarity, the cases $m=2,3,4$ are given explicitly below. For $m \geqslant 5$, the two starter cycles are:

$$(a_0, A_0, b_1, B_1, a_2, A_2, b_3, B_3, \ldots, a_{2m-2}, A_{2m-2}, b_{2m-1}, B_{2m-1}, c_{2m}, C_{2m}),$$

$$(a_0, B_{2m}, c_{2m-2}, A_0, d_1, B_2,\ a_3, C_4, b_5, D_{2m-3}, a_{2m-4}, C_6, b_7, D_{2m-5},$$

$$a_{2m-6}, C_8, b_9, D_{2m-7}, \ldots, a_{2m-2i}, C_{2i+2}, b_{2i+3}, D_{2m-2i-1}, \ldots, a_4, C_{2m-2}, b_{2m-1} D_3, a_2, C_{2m-1}, b_{2m}, D_1).$$

In the second cycle above, i can take any value between 2 and $m-2$ inclusive.

When $m=2$, the second starter cycle is $(a_0, B_4, c_2, A_0, d_1, B_2,\ a_3, C_3, b_4, D_1)$.

When $m=3$, the second starter cycle is

$$(a_0, B_6, c_4, A_0, d_1, B_2, a_3, C_4, b_5, D_3, a_2, C_5, b_6, D_1).$$

When $m=4$, the second starter cycle is

$$(a_0, B_8, c_6, A_0, d_1, B_2, a_3, C_4, b_5, D_5, a_4, C_6, b_7, D_3, a_2, C_7, b_8, D_1).$$

It is an easy exercise to verify that in all cases these eight cycles precisely cover sixteen 1-factors involving edges $x_i Y_j$, $x \in \{a,b,c,d\}$ and $Y \in \{A,B,C,D\}$. □

Lemma 3.4. *There exists a* $K_{2m+1,2m+1}$*-design of order* $4(2m+1)^2+1$.

Proof. Let $v = 4(2m+1)^2+1$ and $V = \mathbb{Z}_v$. Then a $K_{2m+1,2m+1}$-design (V,B) of order v contains $2v$ blocks. We give two starter blocks for B, which are cycled modulo $v = 4(2m+1)^2+1$:

$$\{0,1,2,\ldots,2m : 2m+1, 2(2m+1), 3(2m+1), \ldots, (2m+1)^2\},$$

$$\{0,1,2,\ldots,2m : (2m+1)(2m+2), (2m+1)(2m+3), (2m+1)(2m+4), \ldots, (2m+1)^2.2\}.$$

It is straightforward to verify that (V,B) is a $K_{2m+1,2m+1}$-design of order $4(2m+1)^2+1$. □

Lemma 3.5. *There exists an extended metamorphosis of the design in Lemma* 3.4 *into a* $(4m+2)$*-cycle system.*

Proof. From Lemma 3.1, we may take a 1-factor from each of the $2v$ blocks. These 1-factors need to be rearranged into v cycles of length $4m+2$ (where $v = 4(2m+1)^2+1$).

The 1-factor we take from each starter block is the obvious one:

$$[0,1,2,\ldots,2m : 2m+1, 2(2m+1), 3(2m+1), \ldots, (2m+1)^2],$$

$$[0,1,2,\ldots,2m : (2m+1)(2m+2), (2m+1)(2m+3), (2m+1)(2m+4), \ldots, (2m+1)^2.2].$$

By considering differences modulo $v = 4(2m+1)^2+1 = 16m^2+16m+5$, we can arrange these 1-factors to obtain one starter cycle $(\text{mod}\, v)$ of length $4m+2$. In order to do this, we first consider the differences obtained from the two starter 1-factors above. We list these in Table 1.

Note that the differences labelled $_{(i)}$, $_{(i+1)}$, for each 1-factor, differ by $2m$. However, there is an odd number of differences in each 1-factor list, so we remove the differences marked $*$ and $\bullet$ noting that (with signs attached) they sum to zero or v (which is $16m^2+16m+5$). These observations enable us to write down a $(4m+2)$-cycle, which precisely encompasses all these differences, and so which is a starter cycle using all the 1-factor edges. This we do in Table 2. Entries in the cycle need to be taken modulo $v = 16m^2+16m+5$. It is a straightforward but tedious check to verify that each cycle contains $4m+2$ *distinct* entries.

Table 2 gives more values than necessary for small m. When $m=2$, the 1-factors in a $K_{5,5}$ to C_{10} metamorphosis form a starter 10-cycle (mod 101):

$$(0, 9, 39, 52, 94, 31, 77, 60, 55, 21).$$

When $m=3$, the 1-factors in a $K_{7,7}$ to C_{14} metamorphosis form a starter 14-cycle (mod 197):

$$(0, 13, 44, 100, 168, 187, 76, 156, 51, 14, 7, 179, 117, 43).$$

These starter cycles come from appropriate entries in Table 2.

This completes the lemma. □

Table 1
Differences from two starter 1-factors

	First 1-factor		Second 1-factor	
(0)	$2m+1$		$4m^2+6m+2$	
(1)	$4m+1$		$4m^2+8m+2$	
(2)	$6m+1$	$*$	$4m^2+10m+2$	
(3)	$8m+1$		$4m^2+12m+2$	
(4)	$10m+1$		$4m^2+14m+2$	
	$\vdots$		$\vdots$	
$(m-2)$	$2m^2-2m+1$		$6m^2+2m+2$	
$(m-1)$	$2m^2+1$		$6m^2+4m+2$	
(m)	$2m^2+2m+1$		$6m^2+6m+2$	
$(m+1)$	$2m^2+4m+1$		$6m^2+8m+2$	
$(m+2)$	$2m^2+6m+1$		$6m^2+10m+2$	
	$\vdots$		$\vdots$	
$(2m-2)$	$4m^2-2m+1$		$8m^2+2m+2$	$\bullet$
$(2m-1)$	$-(4m^2+1)$	$\bullet$	$8m^2+4m+2$	$*$
$(2m)$	$-(4m^2+2m+1)$	$\bullet$	$8m^2+6m+2$	$*$

4. Extended metamorphosis: $\lambda = 1$

We are now able to combine the decompositions given in the previous section, to obtain our main result.

Theorem 4.1. *There exists a $K_{2m+1,2m+1}$-design of all orders* 1 *modulo* $(4m+2)^2$ *which has an extended metamorphosis into a* $(4m+2)$*-cycle system of the same order.*

Proof. Let the vertex set of a complete graph of order $x(4m+2)^2+1$ be

$$\{\infty\} \cup \{(i,j) \mid 1 \leqslant i \leqslant x(2m+1),\ 1 \leqslant j \leqslant 8m+4\}.$$

On each set of vertices $\{(a,j) \mid 1 \leqslant j \leqslant 8m+4\} \cup \{(b,j) \mid 1 \leqslant j \leqslant 8m+4\}$, where $a = i_1(2m+1)+a_0$ and $b = i_2(2m+1)+b_0$, with $0 \leqslant i_1 < i_2 \leqslant x-1$ and $1 \leqslant a_0, b_0 \leqslant 2m+1$, we use Lemma 3.3 and place a $K_{2m+1,2m+1}$-decomposition of $K_{8m+4,8m+4}$ which has an extended metamorphosis into a $(4m+2)$-cycle system.

Then on each set of vertices

$$\{\infty\} \cup \{(i,j) \mid w(2m+1)+1 \leqslant i \leqslant (w+1)(2m+1),\ 1 \leqslant j \leqslant 8m+4\},$$

for $0 \leqslant w \leqslant x-1$, we place a $K_{2m+1,2m+1}$-design of order $(4m+2)^2+1$ (see Lemmas 3.4 and 3.5).

The result is an extended metamorphosis as required, proving the theorem. □

5. Extended metamorphosis when $\lambda = 2$

Lemma 5.1. *There exists an extended metamorphosis of* $2K_{2m+1,2m+1}$ *into a* 2*-fold* $(4m+2)$*-cycle system.*

Proof. Let the vertex set of $2K_{2m+1,2m+1}$ be $\{i_a \mid 0 \leqslant i \leqslant 2m\} \cup \{i_b \mid 0 \leqslant i \leqslant 2m\}$. From Lemma 3.1, we may remove $2m$ cycles of length $(4m+2)$, leaving two 1-factors, which we may choose as $[0_a, 1_a, 2_a, \ldots, (2m)_a : 0_b, 1_b, 2_b, \ldots, (2m)_b]$ and $[0_a, 1_a, 2_a, \ldots, (2m-1)_a, (2m)_a : 1_b, 2_b, 3_b, \ldots, (2m)_b, 0_b]$. These two 1-factors combine into the cycle: $(0_a, 0_b, (2m)_a, (2m_b), (2m-1)_a, (2m-1)_b, \ldots, 1_a, 1_b)$. □

Theorem 5.2. *There exists a* 2*-fold $K_{2m+1,2m+1}$-design of any order congruent to* 1 *modulo* $(2m+1)^2$, *which has an extended metamorphosis into a* 2*-fold* $(4m+2)$*-cycle system.*

Proof. Let $v = x(2m+1)^2+1$, and let the vertex set of K_v be

$$\{\infty\} \cup \{(i,j) \mid 1 \leqslant i \leqslant x(2m+1),\ 1 \leqslant j \leqslant 2m+1\}.$$

Table 2
A starter $(4m+2)$-cycle (with differences), modulo $v=(4m+2)^2+1$

	Difference		Cycle entry (in order)
	0		
(1)	$+(4m+1)$		$4m+1$
(2)	$+(10m+1)$		$14m+2$
(3)	$+(14m+1)$		$28m+3$
(4)	$+(18m+1)$		$46m+4$
	$\vdots$		$\vdots$
(i)	$+((4i+2)m+1)$		$2m(i^2+2i-1)+i$
	$\vdots$		$\vdots$
$(m-2)$	$+(4m^2-6m+1)$		$2m^3-4m^2-m-2$
$(m-1)$	$+(4m^2-2m+1)$		$2m^3-3m-1$
(m)	$+(4m^2+6m+2)$		$2m^3+4m^2+3m+1$
$(m+1)$	$+(4m^2+10m+2)$		$2m^3+8m^2+13m+3$
$(m+2)$	$+(4m^2+14m+2)$		$2m^3+12m^2+27m+5$
	$\vdots$		$\vdots$
$(m+j)$	$+(4m^2+(6+4j)m+2)$		$2m^3+4(j+1)m^2+m(2j^2+8j+3)+(2j+1)$
	$\vdots$		$\vdots$
$(2m-2)$	$+(8m^2-2m+2)$		$8m^3-4m^2-3m-3$
	Then use five of the six differences marked $*$ and $\bullet$		
$(2m-1)$	$+(6m+1)$	$*$	$8m^3-4m^2+3m-2$
$(2m)$	$+(8m^2+4m+2)$	$*$	$8m^3+4m^2+7m$
$(2m+1)$	$+(8m^2+2m+2)$	$\bullet$	$8m^3+12m^2+9m+2$
$(2m+2)$	$+(8m^2+6m+2)$	$*$	$8m^3+20m^2+15m+4$
$(2m+3)$	$-(4m^2+1)$	$\bullet$	$8m^3+16m^2+15m+3$
	Then use all the remaining negative differences		
$(2m+4)$	$-(2m+1)$		$8m^3+16m^2+13m+2$
$(2m+5)$	$-(8m+1)$		$8m^3+16m^2+5m+1$
$(2m+6)$	$-(12m+1)$		$8m^3+16m^2-7m$
	$\vdots$		$\vdots$
$(2m+k+5)$	$((8+4k)m+1)$		$8m^3+16m^2-m(2k^2+10k-5)-(k-1)$
	$\vdots$		$\vdots$
$(3m+2)$	$-(4m^2-4m+1)$		$6m^3+18m^2+16m+4\equiv 6m^3+2m^2-1$
$(3m+3)$	$-(4m^2+8m+2)$		$6m^3+14m^2+8m+2$
$(3m+4)$	$-(4m^2+12m+2)$		$6m^3+10m^2-4m$
$(3m+5)$	$-(4m^2+16m+2)$		$6m^3+6m^2-20m-2$
	$\vdots$		$\vdots$
$(3m+w+4)$	$-(4m^2+(12+4w)m+2)$		$6m^3+m^2(10-4w)-m(2w^2+14w+4)-2w$
	$\vdots$		$\vdots$
$(4m+1)$	$-(8m^2+2)$		$20m^2+18m+6$
$(4m+2)$	$-(4m^2+2m+1)$	$\bullet$	$16m^2+16m+5\equiv 0$

First we deal with the case $x=1$. A 2-fold $K_{2m+1,2m+1}$-design of order $v_1=(2m+1)^2+1$ is given by the one starter block modulo v_1:

$$\{0,1,2,\dots,2m : 2m+1, 2(2m+1), 3(2m+1),\dots,2m(2m+1),(2m+1)^2\}.$$

We now give the extended metamorphosis of this 2-fold design of order v_1.

From each of the v_1 blocks, as described in Lemma 3.1, we can take m cycles of length $4m+2$ with a 1-factor remaining. Suppose the v_1 remaining 1-factors are given by the one starter (mod v_1):

$$[0,1,2,\dots,2m : 2m+1, 2(2m+1), 3(2m+1),\dots,2m(2m+1),(2m+1)^2].$$

Table 3
A starter $(4m+2)$-cycle (with differences), modulo $v_1=(2m+1)^2+1$, taken for $v_1/2$ values

	Difference	Cycle entry $(\mathrm{mod}\, v_1 = 4m^2+4m+2)$
(0)		0
(1)	$2m+1$	$2m+1$
(2)	$4m+1$	$6m+2$
(3)	$6m+1$	$12m+3$
	$\vdots$	$\vdots$
(i)	$2mi+1$	$i(i+1)m+i$
	$\vdots$	$\vdots$
(m)	$2m^2+1$	m^3+m^2+m
$(m+1)$	$2m+1$	m^3+m^2+3m+1
$(m+2)$	$4m+1$	m^3+m^2+7m+2
	$\vdots$	$\vdots$
$(m+i)$	$2mi+1$	$m^3+m^2+m(i^2+i+1)+i$
	$\vdots$	$\vdots$
$(2m)$	$2m^2+1$	$2m^3+2m^2+2m$
$(2m+1)$	$2m^2+2m+1$	$2m^3-1$[a]
$(2m+2)$	$-(2m^2+1)$	$2m^3-2m^2-2$
$(2m+3)$	$-(2m^2-2m+1)$	$2m^3-4m^2+2m-3$
	$\vdots$	$\vdots$
$(3m-i+2)$	$-(2mi+1)$	$m^3-m^2+m(i^2-i-1)+i-2$
	$\vdots$	$\vdots$
$(3m+1)$	$-(2m+1)$	m^3-m^2-m-1
$(3m+2)$	$-(2m^2+1)$	m^3+m^2+3m
	$\vdots$	$\vdots$
$(4m-i+2)$	$-(2mi+1)$	$2m^2+m(i^2-i+2)+i$
	$\vdots$	$\vdots$
$(4m)$	$-(4m+1)$	$2m^2+4m+2$
$(4m+1)$	$-(2m+1)$	$2m^2+2m+1=v_1/2$

[a]Note that when m is odd, this value, $2m^3-1$, in the cycle is congruent to $m\,(\mathrm{mod}\, v_1)$, while if m is even, it is congruent to $2m^2+3m+1\,(\mathrm{mod}\, v_1)$.

The $2m+1$ differences $(\mathrm{mod}\, v_1)$ from these $2m+1$ edges we use to form one starter cycle of length $4m+2$ which is taken $\mathrm{mod}\, v_1$ for $v_1/2$ values only (noting that $v_1/2$ is $2m^2+2m+1$).

We give small m first. When $m=2$, thirteen 10-cycles come from the starter $(0,5,14,19,2,15,6,1,18,13)\,(\mathrm{mod}\, 26)$. Note that the differences here are $5,9,5,9,13,-9,-5,-9,-5,-13$. When $m=3$, 25 14-cycles come from the starter $(0,7,20,39,46,9,28,3,34,21,14,45,32,25)$. The differences here are $7,13,19,7,13,19,25,-19,-13,-7,-19,-13,-7,-25$.

We give the general case in Table 3.

Now for the general construction, with $x \geqslant 1$, take:

(i) the blocks $\{\{(a,j)\mid 1\leqslant j\leqslant 2m+1\}:\{(b,j)\mid 1\leqslant j\leqslant 2m+1\}\}$, where $a=i_1(2m+1)+a_0$ and $b=i_2(2m+1)+b_0$, with $0\leqslant i_1<i_2\leqslant x-1$ and $1\leqslant a_0,b_0\leqslant 2m+1$;

(ii) the blocks in a 2-fold design of order $(2m+1)^2+1$ on the vertex set

$$\{\infty\}\cup\{(i,j)\mid w(2m+1)+1\leqslant i\leqslant (w+1)(2m+1),\ 1\leqslant j\leqslant 2m+1\},$$

for each w with $0\leqslant w\leqslant x-1$.

The blocks in (i) above have an extended metamorphosis as described in Lemma 5.1, while the 2-fold design in (ii) above also has an extended metamorphosis, given above.

This completes the theorem. □

6. Concluding remarks

Further spectrum values with higher values of λ (multiples of $2m + 1$) remain open.

Note that the extended metamorphosis from a $K_{t,t}$-design into a $2t$-cycle system yields a cycle system with the interesting property that the majority of the cycles cluster in sets of $\lfloor t/2 \rfloor$ which lie on the same set of $2t$ vertices.

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